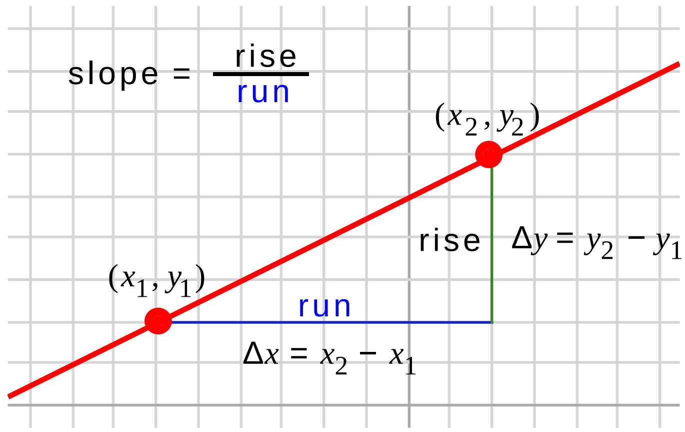
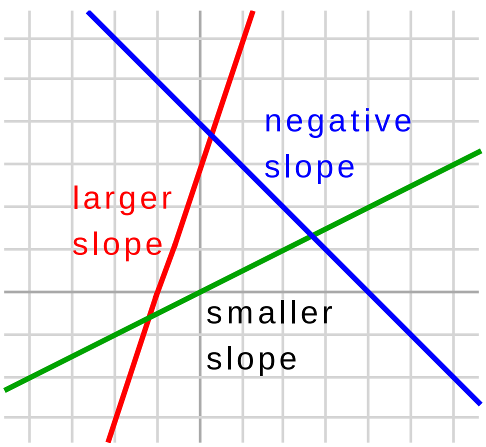
3.1 Slopes of Curves; Derivatives

The slope of a line is a measure of its "steepness": for any two points on the line, it is the ratio of the "rise" (difference in the y − coordinates) to the "run" (difference in the x − coordinates), i.e. $\text{slope} = \Delta y / \Delta x$.

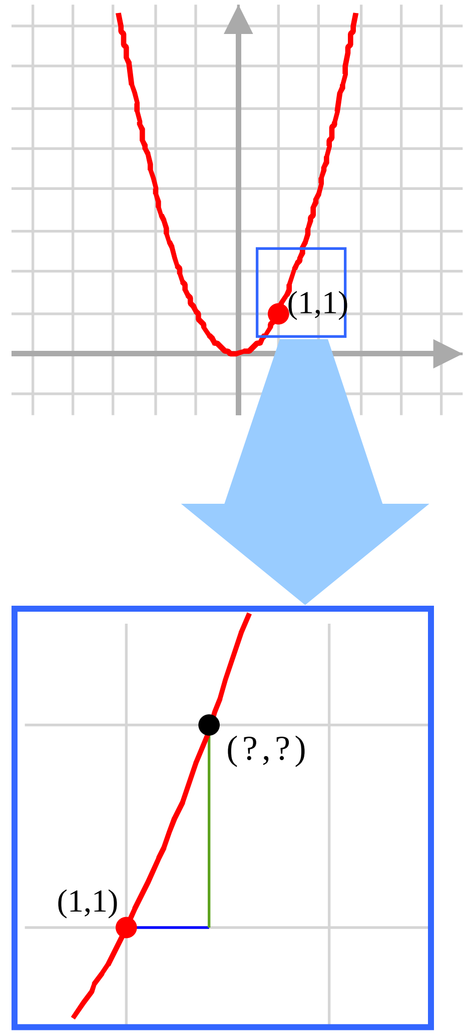




Steep lines have large slopes (large Δy relative to Δx), flatter lines have smaller slopes. Decreasing lines (which go "downhill" as x increases towards the right) have negative slopes (Δx and Δy with opposite signs). The slope of a line doesn't depend on the pair of points on the line used to calculate it; all pairs of points on the same line will give the same slope.

For curves that aren't lines, the idea of a single overall slope is not very useful. Intuitively, the steepness of a typical curve is different at different places on the curve, so an appropriate definition of slope for the curve should somehow reflect this variable steepness.

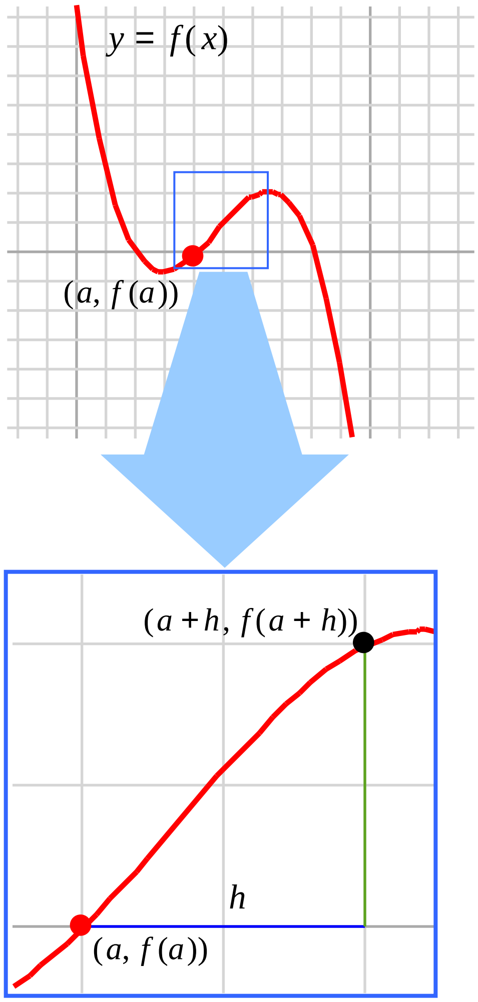
Let's look at how we could define the slope of a particular curve, say $y = x^2$, near a typical point on the curve, say $(1,1)$.



Use your grapher to plot the curve and the point, and investigate by zooming in closely on the point. Notice what happens as you zoom: as you get closer to the point, the visible part of the curve gets progressively straighter, and eventually becomes indistinguishable from a straight line. Looked at "up close", the part of the curve $y = x^2$ near $(1,1)$ is approximately a straight line, so it makes sense to think of the slope of $y = x^2$ near $(1,1)$ as the slope of this approximate line. Read off from your zoomed graph the coordinates of another point on the curve and use it with the point $(1,1)$ to calculate a slope. You should get a number very close to 2 – in fact, the closer in you zoom, the nearer this calculated slope should be to 2.

✔ Find the approximate slope of $y = x^2$ near $(2,4$ by zooming.

Now let's do this same process algebraically, for the graph of a general function $y = f(x)$ at a general point $(a, f(a))$ on it. Suppose we zoom in on $(a, f(a))$ until we're satisfied that the visible part of the curve is nearly a line.



We can calculate the slope of this approximate line from the given point $(a, f(a))$ and a nearby point on the curve, say with $x = a + h$ for some small value h and $y = f(a+h)$:

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

Now suppose we zoom in closer and closer to $(a, f(a))$.The visible part of the curve gets closer and closer to a straight line as we do so. At each step, to calculate the slope of this "improved" line, we need to take a new nearby point $(a+h, f(a+h))$ even closer to $(a, f(a))$ than before, i.e. we have to take h smaller and smaller. In effect, we are performing a limiting process here: we are looking at the value as $h \to 0$ of our above expression for the slope near $(a, f(a))$. It thus makes sense to define the slope of the curve *at* $(a, f(a))$ to be the limit of this expression.

Definition. The ***slope of a curve*** $y = f(x)$ at the point $(a, f(a))$ on it is defined to be the number

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Example. Let's use this limit to check our earlier experimental result that the slope of the curve $y = x^2$ at the point $(1,1)$ is 2. Here $a = 1$ and $f(x) = x^2$, so we have

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h}$$

$$= \lim_{h \to 0} \frac{2h + h^2}{h}$$

$$= \lim_{h \to 0} (2 + h)$$

$$= 2$$

as expected. ■

Example. We find the slope of the curve $y = \sqrt{x}$ at the point $(4, 2.$ Here $a = 4$ and $f(x) = \sqrt{x}$, so the slope is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} \times \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2}$$

$$= \lim_{h \to 0} \frac{(4+h) - 4}{h(\sqrt{4+h} + 2)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2}$$

$$= \frac{1}{\sqrt{4} + 2}$$

$$= \tfrac{1}{4}$$

Check this answer by zooming in on the graph of $y = \sqrt{x}$ near $(4, 2.$ ■

✔ Find the slope of the curve $y = \frac{1}{x}$ at $(2, \frac{1}{2})$ and check your answer by zooming in on the graph.

The special limit used to find slopes of curves occurs in many other contexts, and so has a name and a notation.

Definition. For any function $y = f(x)$, the number

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if it exists, is called the ***derivative*** of the function f at the value $x = a$.

Example. Let's use the definition to calculate the derivative of $f(x) = (x+3)^2$ at $x = 0$.

$$
\begin{aligned}
f'(0) &= \lim_{h \to o} \frac{f(0+h) - f(0)}{h} \\
&= \lim_{h \to o} \frac{(0+h+3)^2 - (0+3)^2}{h} \\
&= \lim_{h \to o} \frac{(h^2 + 6h + 9) - (9)}{h} \\
&= \lim_{h \to o} (h+6) \\
&= 6 \qquad \blacksquare
\end{aligned}
$$

✔ Use the definition to find the derivative of $f(x) = \sqrt{x}(x-2)$ at $x = 2$.

✔ Find the derivative of $\sin x$ at $x = 0$ and the derivative of $\cos x$ at $x = \pi$.

✔ Identify "by inspection" a function f and a number a such that

$$
f'(a) = \lim_{h \to 0} \frac{(2+h)e^{(2+h)} - 2e^2}{h}.
$$

Problem. Calculate the derivative of $f(x) = \frac{x}{x+1}$ at $x = 2$ from the definition.

$$
\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h \to 0} \frac{\frac{2+h}{(2+h)+1} - \frac{2}{2+1}}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \left\{ \frac{2+h}{3+h} - \frac{2}{3} \right\} \\
&= \lim_{h \to 0} \frac{1}{h} \left\{ \frac{3(2+h) - (3+h)2}{(3+h)(3)} \right\} \\
&= \lim_{h \to 0} \frac{1}{h} \left\{ \frac{h}{3(3+h)} \right\} \\
&= \lim_{h \to 0} \frac{1}{3(3+h)} \\
&= \tfrac{1}{3(3)} \quad \blacksquare
\end{aligned}
$$

✔ Use the definition to calculate the derivative of $f(x) = \frac{1}{\sqrt{x+1}}$ at $x = 3$.

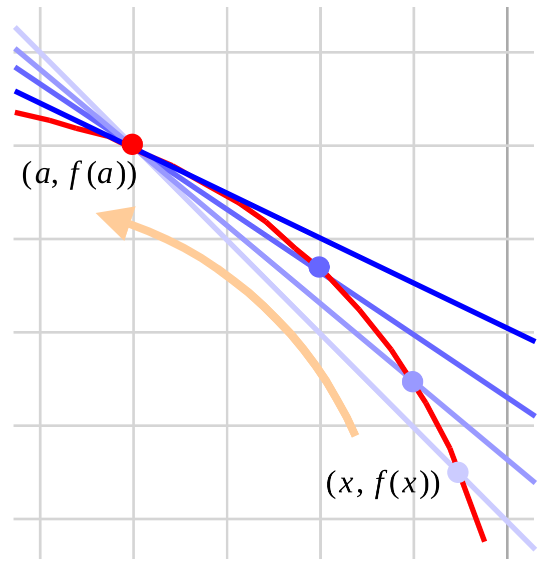
✔ To calculate derivatives of more complicated functions, you can often use your CAS to do the necessary work. Use your CAS to find the derivative of

$$
f(x) = \frac{x^2 - 5}{x^2 + 1}
$$

at $x = 4$ from the definition.

3.2 Tangent Lines and Linear Approximations

Intuitively, a line is tangent to a curve at some point on it if the line "touches" the curve at that point. We can think of the tangent line as the limiting position of a line through the point and another nearby point on the curve as the second point approaches the first.



Let's use this idea to find the slope of the tangent line and then its equation. Suppose we are given a point $(a, f(a))$ on the curve $y = f(x)$. If $(x, f(x))$ is a nearby point on the curve, then the slope of the line joining the two points is

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}.$$

The slope of the tangent line at $(a, f(a))$ is the limit of this expression as $(x, f(x))$ approaches $(a, f(a))$ along the curve, i.e. as x approaches a:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

If we set $h = x - a$, then the statement "$x \to a$" is equivalent to "$h \to 0$", so we can write the limit as

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

i.e. we have $f'(a)$.

> The slope of the line tangent to the graph of $y = f(x)$ at the point $(a, f(a))$ on it is $f'(a)$.

Example. To find the equation of the line tangent to $y = x^3$ at the point $(2,8)$ on it, we set $f(x) = x^3$ and calculate the necessary slope $f'(2)$:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{(2+h)^3 - 2^3}{h} \\ &= \lim_{h\to 0} \frac{(8+12h+6h^2+h^3)-8}{h} \\ &= \lim_{h\to 0}(12+6h+h^2) \\ &= 12 \end{aligned}$$

To find the tangent line, equate this slope to the slope from the points $(2,8)$ and (x,y) and solve for y:

$$\begin{aligned} \frac{y-8}{x-2} &= 12, \\ y-8 &= 12(x-2) \\ &= 12x-24, \\ y &= 12x-16. \quad \blacksquare \end{aligned}$$

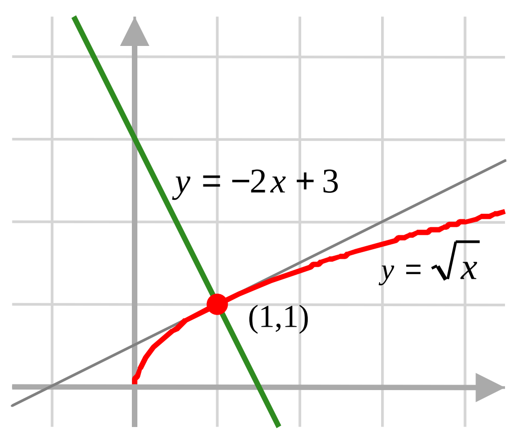
Example. The normal line to a curve at a point on it is the line perpendicular to the curve at that point, i.e. perpendicular to its tangent line at that point. (Remember that two lines are perpendicular if their slopes are negative reciprocals of each other.)

To find the normal line to the curve $y = 1/\sqrt{x}$ at $(1,1)$, we first find the slope of the tangent line at $(1,1)$. For $f(x) = 1/\sqrt{x}$,

$$
\begin{aligned}
f'(4) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h \to 0} \frac{1/\sqrt{1+h} - 1/\sqrt{1}}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \left\{ \frac{1}{\sqrt{1+h}} - 1 \right\} \\
&= \lim_{h \to 0} \frac{1}{h} \left\{ \frac{1 - \sqrt{1+h}}{\sqrt{1+h}} \right\} \\
&= \lim_{h \to 0} \frac{1}{h} \left\{ \frac{1 - \sqrt{1+h}}{\sqrt{1+h}} \times \frac{1 + \sqrt{1+h}}{1 + \sqrt{1+h}} \right\} \\
&= \lim_{h \to 0} \frac{1}{h} \left\{ \frac{1 - (1-h)}{\sqrt{1+h}\,(1 + \sqrt{1+h}\,)} \right\} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{1+h}\,(1 + \sqrt{1+h}\,)} \\
&= \tfrac{1}{2}
\end{aligned}
$$

This is the slope of the tangent line through $(1,1)$, so the slope of the normal line through $(1,1)$ is -2. The equation of the normal line is then

$$\frac{y-1}{x-1} = -2$$



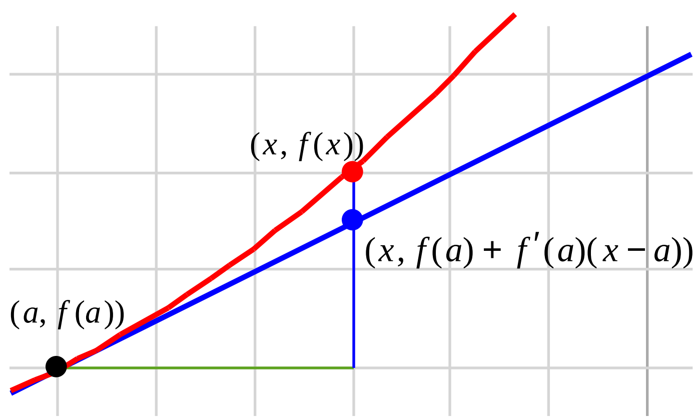
which simplifies to $y = -2x + 3$. ■

✔ Find equations for the tangent line and normal line to the parabola $y = x^2 + 4x$ at the origin.

✔ Find the derivative of $f(x) = e^x$ at $x = 0$. (Hint: how was the number e defined?)

At any point of on the graph of a function, the tangent line has the derivative of the function at that point as slope, i.e. the tangent line has the same slope as the curve at that point. This means that when we zoom in on the point, the line that the curve "straightens out" into is in fact this same tangent line. Since the curve then gets closer and closer to its tangent line the closer we zoom in on the point of tangency, we can use the tangent line to approximate the curve near that point.

The line tangent to the curve $y = f(x)$ at $(a, f(a))$ has slope $f'(a)$, and so has equation



$$\frac{y - f(a)}{x - a} = f'(a),$$

or equivalently,

$$y = f(a) + f'(a)(x - a).$$

Near $x = a$, the y-coordinate for the curve is approximately the y-coordinate for the line.

Definition. The ***linear approximation*** near $x = a$ for the function $y = f(x)$ is

$$f(x) \approx f(a) + f'(a)(x - a).$$

Example. We find the linear approximation for the parabola $y = x^2 - 5x + 3$ for values of x near 1. For $a = 1$ and $f(x) = x^2 - 5x + 3$ the approximation is $f(x) \approx f(1) + f'(1)(x - 1)$. We have $f(1) = -1$ and we need to find $f'(1)$:

$$f'(1) = \lim_{h \to 0} \frac{\left\{(1+h)^2 - 5(1+h) + 3\right\} - \{-1\}}{h}$$
$$= \lim_{h \to 0} (h - 3) = -3$$

Then near $x = 1$, $f(x) \approx -1 + (-3)(x - 1)$, i.e., $x^2 - 5x + 3 \approx -3x + 2$. ■

✔ Use a linear approximation near $x = 0$ to show that $\sin x \approx x$ for small values of x. About how small must x be to ensure accuracy to 2 decimal places? (Experiment by tabulating some values of x vs. $\sin x$.)

Example. We use a linear approximation to estimate $\sqrt{83}$. The function in this case is $f(x) = \sqrt{x}$, and the nearest value to 83 of an easy-to-find square is $a = 81$. The approximation is $f(83) \approx f(81) + f'(81)(3 - 1)$, so we need

$$
\begin{aligned}
f'(81) &= \lim_{h \to 0} \frac{\sqrt{81+h} - \sqrt{81}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{81+h} - 9}{h} \times \frac{\sqrt{81+h} + 9}{\sqrt{81+h} + 9} \\
&= \lim_{h \to 0} \frac{(81+h) - 81}{h\sqrt{81+h} + 9} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{81+h} + 9} \\
&= \tfrac{1}{18}
\end{aligned}
$$

Then $\sqrt{83} \approx \sqrt{81} + \frac{1}{18}(83-81) = 9\frac{1}{9} \approx 9.1111$ (correct to 4 decimal places). The actual value of $\sqrt{83}$ is 9.1104 (to 4 decimal places). ■

✔ Use a linear approximation to estimate $\frac{1}{2.003}$.

==

In economics, derivatives of quantities such as cost, revenue, etc. which are functions of the number of units produced are called marginal quantities. For example, if $C(x)$ is the cost of producing x units of some commodity, then $C'(a)$ is the marginal cost of producing a units. For values of x close to a, linear approximation of $C(x)$ gives

$$C(x) \approx C(a) + C'(a)(x-a).$$

Since only whole numbers of units can be produced, x and a must be positive integers. If $x = a+1$, we have $C(a+1) \approx C(a) + C'(a) \times 1$, i.e.

$$C'(a) \approx C(a+1) - C(a)$$

(assuming that when a large number x of units is produced, the extra cost for one more unit is relatively small). This says that the marginal cost is approximately the difference in cost between producing the $(a+1)^{\text{st}}$ unit and the a^{th} unit – the extra cost of producing the $(a+1)^{\text{st}}$ unit. Similarly, for a revenue function R, marginal revenue, $R'(a)$ approximates the additional revenue received from selling the $(a+1)^{\text{st}}$ unit.

Example. Suppose that the cost (in dollars) of producing x widgets is $C(x) = 13 + x + \frac{1}{5}x^2$ and that all will be sold if the selling price is $p(x) = \frac{2}{5}(25 - x)$. The marginal cost of producing the 10^{th} unit is

$$\begin{aligned}
C'(10) &= \lim_{h \to 0} \frac{C(10+h) - C(10)}{h} \\
&= \lim_{h \to 0} \frac{\left\{13 + (10+h) + \frac{1}{5}(10+h)^2\right\} - \left\{13 + 10 + \frac{1}{5}10^2\right\}}{h} \\
&= \lim_{h \to 0}(5 + \tfrac{1}{5}h) \\
&= 5
\end{aligned}$$

To compare, the actual cost of producing the 10^{th} unit is $C(10) - C(9) = 4.8$. The revenue function (selling price times number sold) is

$$R(x) = x \times \tfrac{2}{5}(25 - x) = 10x - \tfrac{2}{5}x^2$$

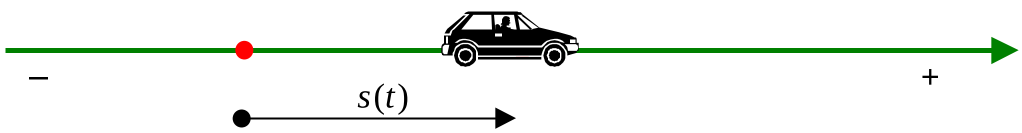
The marginal revenue on the 10^{th} unit is

$$\begin{aligned}
R'(10) &= \lim_{h \to 0} \frac{\left\{10(10+h) - \frac{2}{5}(10+h)^2\right\} - \left\{10(10) - \frac{2}{5}(10)^2\right\}}{h} \\
&= \lim_{h \to 0}(2 - \tfrac{2}{5}h) \\
&= 2
\end{aligned}$$

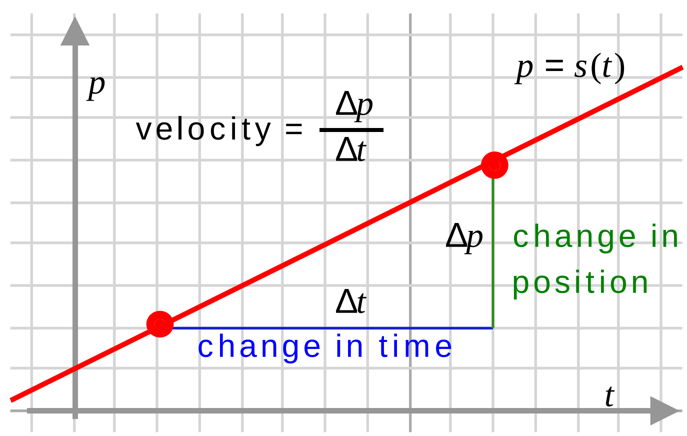
i.e. selling 10 units instead of 9 produces approximatly \$2 extra revenue. ■

3.3 Velocity and Other Rates of Change

Suppose we want to describe the motion of an object moving along a straight path, say a car along a road. If we pick some reference point on the path and a positive direction along the path, the position p of the object relative to this point at any time t is its "signed distance" from the reference point and depends on t, i.e. $p = s(t)$ for some function s. ("Signed distance" means that the sign of $s(t)$ is positive whenever the object is on the positive side of the reference point and negative when it is on the negative side of this point.)

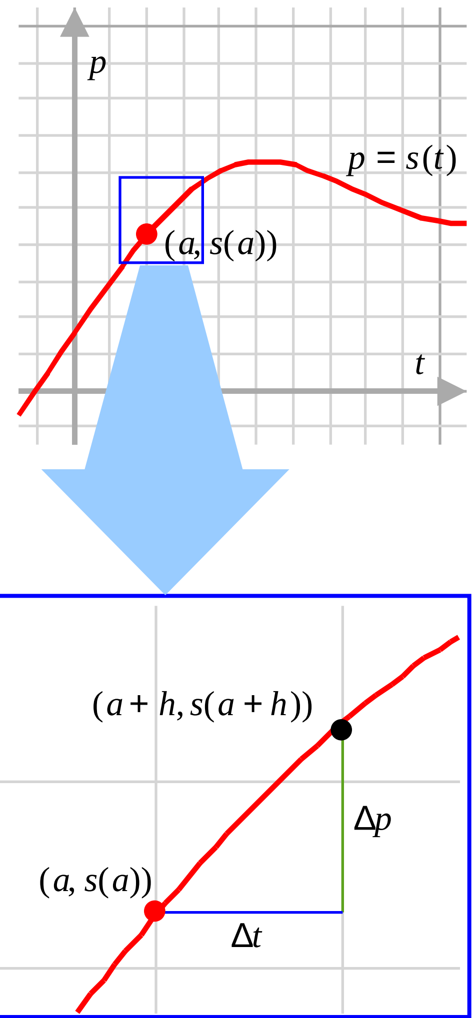


Suppose first that the graph of the function s is a straight line, i.e. suppose that it has constant slope. This slope gives the change in postion (Δp) for a corresponding change in time (Δt), i.e. it gives the velocity of the object – the reading on the car's speedometer.



A linear position vs time curve thus represents an object moving at a constant velocity given by the slope of this line. Note that if the velocity is negative (if Δp is negative while Δt is positive), the object is moving in the negative direction along its path.

What if the position vs time curve is not a straight line? Now the object is moving with variable velocity - the car's speedometer reading is changing. Of course, over very small intervals of time, it doesn't change much - for a very small change in time, the velocity is nearly constant. To estimate this approximately constant velocity near any instant $t = a$, we can look at the position vs. time curve for a very small time interval near $t = a$. Suppose we zoom in on the point $(a, s(a))$ until the curve appears straight, i.e. until its slope appears constant. Then we can calculate the object's approximate velocity near time $t = a$ as we did with the linear distance vs time curve: the approximate velocity near $t = a$ is the slope of this near-line:



$$\begin{matrix}\text{approximate}\\ \text{velocity}\end{matrix} = \text{slope} = \frac{\Delta p}{\Delta t} = \frac{s(a+h)-s(a)}{h}.$$

Intuitively, then, the speedometer reading *at* instant $t = a$ should be the limit of this expression as the length $\Delta t = h$ of the time interval approaches 0, i.e. it should be $s'(a)$, the slope of the position vs. time curve at instant $t = a$.

Defintion. If an object moves along a straight path so that its position relative to some reference point on that path is $p = s(t)$ at time t, then its ***instantaneous velocity*** at time $t = a$ is defined to be $s'(a)$, if this derivative exists.

Example. A car moves along a straight road so that its distance from its starting point at time t hours past noon is $10t^2 + 60t$ km. Then if $s(t)$ is the position of the car at time t relative to its starting point, $s(t) = 10t^2 + 60t$, and the car's velocity at 1 p.m. is

$$\begin{aligned} s'(1) &= \lim_{h\to 0} \frac{s(1+h) - s(1)}{h} \\ &= \lim_{h\to 0} \frac{\{10(1+h)^2 + 60(1+h)\} - 70}{h} \\ &= \lim_{h\to 0}(10h + 80) \\ &= 80 \end{aligned}$$

The units are units of $s(t)$ divided by units of t, i.e. the velocity at 1 p.m. is 80 km/hr. Note that if the car travels for a total of 3 hours, its ***average velocity*** for the trip is

$$\frac{\text{total distance}}{\text{total time}} = \frac{s(3) - s(0)}{3-0} = \frac{270}{3} = 90 \text{ km/hr.} \quad ■$$

Example. After t seconds, a stone dropped off a 100 m cliff has fallen $4.9t^2$ m. We find how fast it is going when as hits the ground.

The postion of the stone relative to the top of the cliff is $s(t) = -4.9t^2$ (the negative sign is there because we normally measure upward distances as positive). We need $s'(a)$, where a is the time it hits the ground. This happens when $s(t) = -100$, i.e. when $-4.9t^2 = -100$, or $t = \frac{10}{7}$ sec. Thus

$$
\begin{aligned}
s'(\tfrac{10}{7}) &= \lim_{h\to 0} \frac{s(\tfrac{10}{7}+h) - s(\tfrac{10}{7})}{h} \\
&= \lim_{h\to 0} \frac{-4.9(\tfrac{10}{7}+h)^2 - \left(-4.9(\tfrac{10}{7})^2\right)}{h} \\
&= \lim_{h\to 0} \frac{-4.9\left\{\tfrac{100}{49} + \tfrac{200}{7}h + h^2 - \tfrac{100}{49}\right\}}{h} \\
&= -4.9 \lim_{h\to 0}\left(\tfrac{200}{7} + h\right) \\
&= -4.9\left(\tfrac{200}{7}\right) = -140.
\end{aligned}
$$

The stone is travelling at 140 m/sec. when it hits the ground. (The minus sign indicates that it is moving downward at the time.) ■

Example. Suppose that a weight at the end of a spring oscillates with ***simple harmonic motion***, i.e suppose that its distance above its starting point at time t is given by

$$s(t) = A\sin\left(\frac{2\pi t}{P}\right),$$

where A and P are positive constants. Since the sine function oscillates between –1 and +1, $s(t)$ oscillates between $-A$ and $+A$, so A is the ***amplitude*** of the oscillation. The sine function undergoes one complete oscillation whenever its argument increases by 2π, i.e. whenever t/P increases by 1, or t increases by P, so P is the ***period*** of the oscillation.

We find the velocity of the weight as it completes its first oscillation, i.e. when $t = P$.

$$
\begin{aligned}
s'(P) &= \lim_{h\to 0}\frac{s(P+h)-s(P)}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left\{A\sin\left(\frac{2\pi(P+h)}{P}\right)-A\sin\left(\frac{2\pi(P)}{P}\right)\right\}\\
&= \lim_{h\to 0}\frac{A}{h}\left\{\sin\left(2\pi+\frac{2\pi h}{P}\right)-\sin(2\pi)\right\}\\
&= A\lim_{h\to 0}\frac{\sin\left(\frac{2\pi h}{P}\right)}{h}\\
&= A\lim_{h\to 0}\frac{\sin\left(\frac{2\pi h}{P}\right)}{\left(\frac{2\pi h}{P}\right)}\frac{2\pi}{P}\\
&= A\times 1\times\frac{2\pi}{P}\\
&= \frac{2\pi A}{P}
\end{aligned}
$$

since sine is periodic and since sin 2 =0

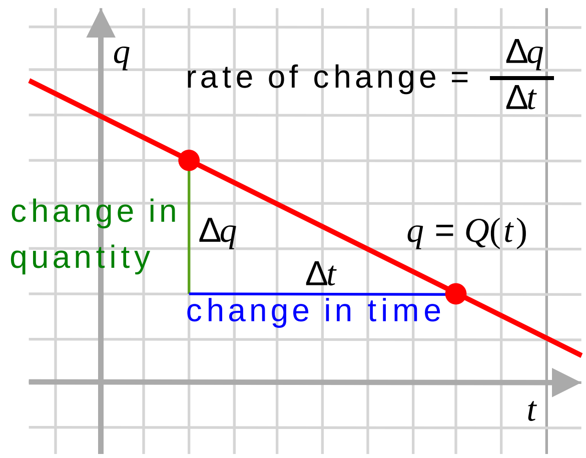
put it in the form $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$

So if $A = 3$ cm and $P = 5$ sec, for example, the velocity is $\frac{6\pi}{5}$ cm/sec, or approximately 3.77 cm/sec.

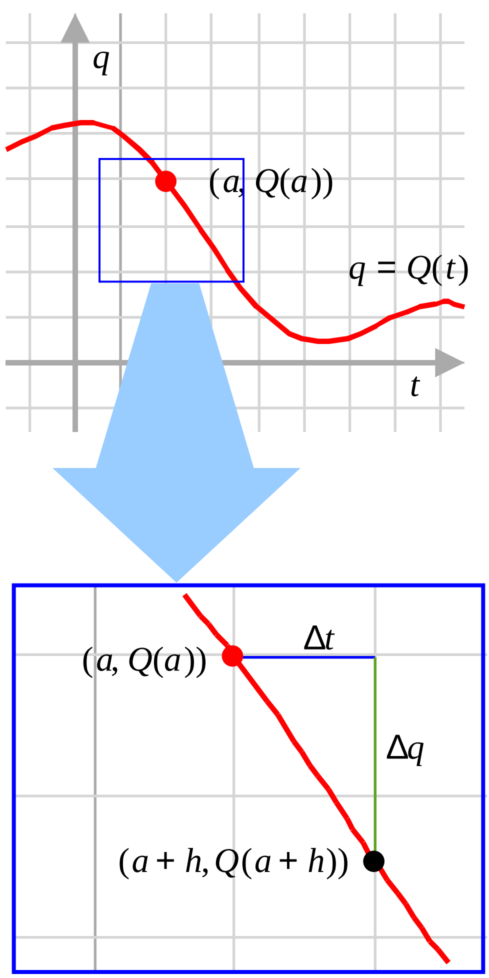
Velocity is a rate of change; the rate of change of position with respect to time. Many other rates of change can be modeled by derivatives; in fact, a rate of change is a derivative's "fundamental nature" – all derivatives are rates of change in some sense.

Consider any quantity q that varies with time – q could be the volume of water in an draining tank, the population of a country, the temperature of a cooling object, the value of an investment or one of many other possible quantities. Since it depends on time t, $q = Q(t)$ for some function Q.

If the graph of Q is a straight line, then its slope is constant, and represents the rate of change of the quantity q (Δq) with respect to the corresponding change in time t (Δt).



If the graph of Q is not a straight line, the rate of change of q with respect to time is not constant. However, for small enough changes in time, this rate of change doesn't vary much and the graph of $q = Q(t)$ doesn't differ much from a straight line. If we zoom in on the curve $q = Q(t)$ near any instant of time $t = a$, the rate of change of q with respect to time near $t = a$ can be approximated by the slope



$$\begin{aligned}\text{approximate rate of change}\\ &= \text{slope}\\ &= \frac{\Delta q}{\Delta t}\\ &= \frac{Q(a+h) - Q(a)}{h}\end{aligned}$$

of this near-linear zoomed curve:

The closer we zoom, the smaller the time interval becomes and the closer this slope approaches the slope of the curve at $(a, s(a))$. It thus makes sense to define the rate of change *at* the instant $t = a$ by the slope of the curve at this point, i.e. by the derivative of $q = Q(t)$ at $t = a$.

> If $q = Q(t)$ represents the value of some quantity that varies with time t, then at time $t = a$, the ***instantaneous rate of change*** of q with respect to time is defined to be $Q'(a)$, if this derivative exists.

✔ If $h = H(t)$ represents the height (in meters) of water in a tank at time t minutes past 4 p.m., what does $H'(3)$ represent (in general terms, in words other than "instantaneous rate of change")? What are the units for $H'(3)$? If $H'(3) = -5$, what is the significance of the negative sign?

Example. A counter top is contaminated with a large and growing population of microbes. The number (in millions) of microbes per square centimeter of the counter top t minutes after it is sprayed with disinfectant is given by

$$9 + 2t - t^{3/2}, \quad 0 \le t \le 9.$$

(So there are 9 million per square cm. to start and none at $t = 9$.) Is the microbe population increasing or decreasing one minute later? four minutes later?

The number of millions of microbes per square cm. t minutes after spraying is

$$N(t) = 9 + 2t - t^{3/2}.$$

The rate of change of this number after 1 minute is $N'(1)$ and after 4 minutes is $N'(4)$; we need to know their signs. Since N is a relatively complicated function, to avoid having to do two potentially complicated limits, let's calculate $N'(a)$ for a general value of a, and then substitute $a = 1$ and $a = 4$.

$$N'(a) = \lim_{h \to 0} \frac{N(a+h) - N(a)}{h}$$

$$= \lim_{h \to 0} \frac{\left\{9 + 2(a+h) - (a+h)^{3/2}\right\} - \left\{9 + 2(a) - a^{3/2}\right\}}{h}$$

$$= \lim_{h \to 0} \left\{ 2 - \frac{(a+h)^{3/2} - a^{3/2}}{h} \right\}$$

$$= 2 - \lim_{h \to 0} \frac{(a+h)^{3/2} - a^{3/2}}{h}$$

$$= 2 - \lim_{h \to 0} \frac{(a+h)^{3/2} - a^{3/2}}{h} \times \frac{(a+h)^{3/2} + a^{3/2}}{(a+h)^{3/2} + a^{3/2}}$$

rationalize

$$= 2 - \lim_{h \to 0} \frac{(a+h)^3 - a^3}{h\left\{(a+h)^{3/2} + a^3\right\}}$$

$$= 2 - \lim_{h \to 0} \frac{(a^3 + 3a^2h + 3ah^2 + h^3) - a^3}{h\left\{(a+h)^{3/2} + a^{3/2}\right\}}$$

expand the cube

$$= 2 - \lim_{h \to 0} \frac{3a^2 + 3ah + h^2}{(a+h)^{3/2} + a^{3/2}}$$

$$= 2 - \frac{3a^2}{2a^{3/2}}$$

$$= 2 - \tfrac{3}{2} a^{1/2}$$

Since $N'(1) = \frac{1}{2} > 0$, the microbe population per square cm. is increasing at time $t = 1$ min., at a rate of half a million per minute. But $N'(4) = -1 < 0$, so after 4 minutes, the microbe population per square cm. is decreasing, at a rate of 1 million per minute.

Plot the function N with your grapher and use the graph to estimate the turn-around point (i.e. the time at which the population reaches its maximum and starts to decrease) and estimate the maximum population. ■

✔ Suppose that the height of the water surface of a river a distance d km. from its mouth is given in meters by $H(d)$. Interpret $H'(5)$. What are the appropriate units for $H'(5)$? Is it positive or negative?

3.4 Derivatives as Functions

Suppose we want to calculate $f'(a)$ for multiple values of a – say we want to look at the velocity of a moving object at several instances of time, for example. Rather than doing several limit calculations at different values of a with the same function, it makes sense to do one calculation at a general unspecified value, say x, and then substitute values of a as necessary. The resulting limit depends on x, i.e. it is a function of x.

Definition For a function f of a variable x, the function f' given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is called the ***derivative*** of f. The process of finding a derivative is called ***differentiation***.

Example. If $f(x) = \sqrt{x}$, then the derivative of f is

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \tfrac{1}{2\sqrt{x}} \end{aligned}$$

So if we want to find the tangent line at some point on $y=\sqrt{x}$, say at $(4,2,$ we use the derivative function to find a numerical slope $f'(4)=1/4$, and then the line is

$$\frac{y-2}{x-4}=\tfrac{1}{4}$$

which simplifies to $x-4y=-4$.

In general, the domain of f' consists of all x in the domain of f for which the derivative is defined. In this case, the domain of f is $[0,\infty)$, but since the derivative is not defined at $x=0$, the domain of f' is $(0,\infty)$. ■

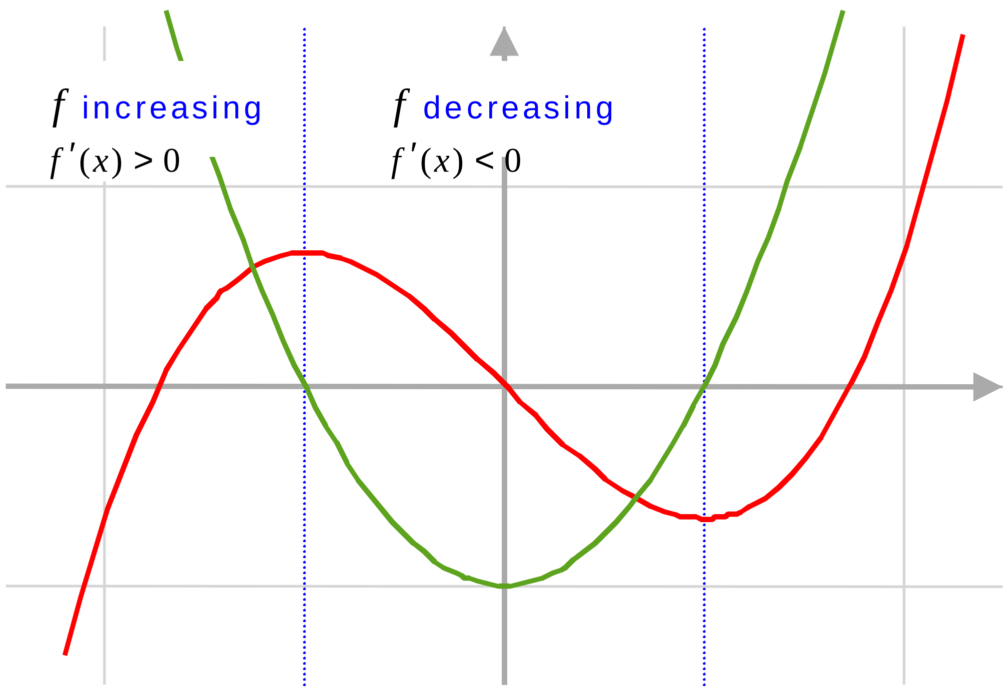
✔ Show that the derivative of the function $f(x)=x^2+x$ is $f'(x)=2x+1$. The curve $y=f(x)$ crosses the *x*-axis at $x=-1$ and $x=0$. Show that the tangent lines to the curve at these points are perpendicular.

At each point along the curve $y=f(x)$, the derivative $f'(x)$ gives the slope of the curve. More accurately, for any *x*, the value of $f'(x)$ (the *y*-coordinate of the curve $y=f'(x)$) gives the slope of the $y=f(x)$.

Example. The derivative of $f(x)=\frac{1}{3}x^3-x$ is

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{\left\{\tfrac{1}{3}(x+h)^3-(x+h)\right\}-\left\{\tfrac{1}{3}x^3-x\right\}}{h}\\
&= \lim_{h\to 0}\frac{\tfrac{1}{3}(x^3+3x^2h+3xh^2+h^3)-x-h-\tfrac{1}{3}x^3+x}{h}\\
&= \lim_{h\to 0}\left(x^2+xh+\tfrac{1}{3}h^2-1\right)\\
&= x^2-1\\
&= (x+1)(x-1)
\end{aligned}
$$

The graph of f' is thus a parabola crossing the x-axis at $x=-1$ and $x=+1$. Here are the graphs of **function** and its **derivative** plotted together on the same axes.



At any value of x, the value of $y = f'(x) = x^2 - 1$ (the y-coordinate of the green curve) gives the slope of the original curve $y = f(x) = \tfrac{1}{3}x^3 - x$ (the red curve). Note that the graph of f is increasing wherever $f'(x) > 0$, i.e.

wherever the graph of f' is above the x-axis, and decreasing wherever $f'(x) < 0$, i.e. wherever the graph of f' is below the x-axis. ■

The conclusions of this example are true in general. A third possibility, $f'(x) \equiv 0$, occurs wherever the rate of change of $f(x)$ is zero, i.e. where $f(x)$ doesn't change.

Information from the sign of the derivative For any function *f*,

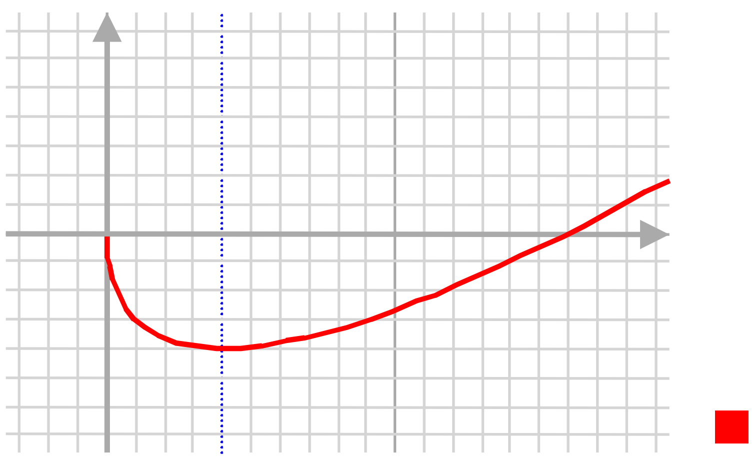
- on intervals where $f'(x) > 0$, f is increasing
- on intervals where $f'(x) < 0$, f is decreasing
- on intervals where $f'(x) \equiv 0$, f is constant.

(A rigorous proof of this statement requires the Mean Value Theorem.)

Example We find intervals on which the function $f(x) = x - 4\sqrt{x}$ is increasing, decreasing or constant. First, we need the derivative.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{\left\{x+h-4\sqrt{x+h}\right\}-\left\{x-4\sqrt{x}\right\}}{h} \\
&= \lim_{h\to 0}\left\{1+\frac{4\sqrt{x}-4\sqrt{x+h}}{h}\right\} \\
&= 1+\lim_{h\to 0}\frac{4\left\{\sqrt{x}-\sqrt{x+h}\right\}}{h}\times\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} \\
&= 1+4\lim_{h\to 0}\frac{x-(x+h)}{h\left(\sqrt{x}+\sqrt{x+h}\right)} \\
&= 1+4\lim_{h\to 0}\frac{-1}{\sqrt{x}+\sqrt{x+h}} \\
&= 1-\frac{2}{\sqrt{x}}
\end{aligned}
$$

The function is increasing wherever $f'(x)>0$, i.e. where $1>\frac{2}{\sqrt{x}}$ or $x>4$. It is decreasing wherever $f'(x)<0$, i.e. where $1<\frac{2}{\sqrt{x}}$ or $x<4$. Here's the graph of $y=f(x)=x-4\sqrt{x}$.



The interpretation of a derivative as a function is an extension of the interpretation a derivative at a point: it is a function giving the rate of change of the original function for any arbitrary values of its independent variable. Thus, for example, if the position of an object on a line relative to some reference point is $p = s(t)$ at time *t*, then $s'(t)$ is a function giving the object's velocity at any arbitrary time *t*.

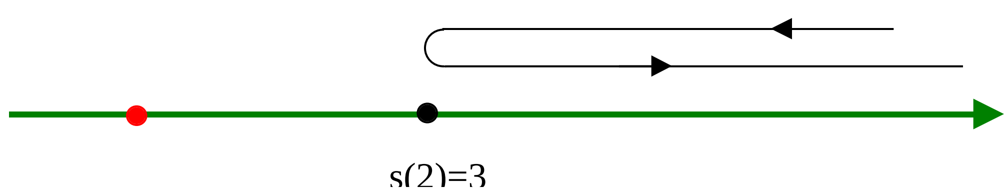
Example Let's analyse the direction of motion of an object that moves along a straight path so that its position relative to some reference point on that path is

$$s(t) = t^2 - 4t + 7$$

meters at time *t* sec. First, we'll find the derivative of *s*.

$$\begin{aligned} s'(t) &= \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} \\ &= \lim_{h \to 0} \frac{\left\{(t+h)^2 - 4(t+h) + 7\right\} - \left\{t^2 - 4t + 7\right\}}{h} \\ &= \lim_{h \to 0} (2t + h - 4) \\ &= 2t - 4. \end{aligned}$$

Then the object is travelling in the negative *x*-direction when $s'(t) < 0$, i.e. when $t < 2$, and in the positive *x*-direction when $s'(t) > 0$, i.e. when $t > 2$. The turn-around point occurs when $t = 2$, when the velocity is 0 - the object is not stopped then, but just changing direction. At this point, its position is $s(2) = 3$.



✔ A point moves upward along the y-axis so that its y-cordinate at time t sec. is $y(t) = t^{3/2}$ units. When is its velocity 3 units/sec. ?

3.5 Higher Derivatives, Concavity and Acceleration

Since the derivative of a function is another function, we can repeat the differentiation process to find the ***second derivative***, the ***third derivative***, and higher order derivatives. These repeated derivatives are denoted by f'', f''', etc., or by $f^{(n)}$ for the n^th derivative of f. The function f itself is sometimes taken to be the 0^th derivative: $f^{(0)} \equiv f$.

Example. We find the second derivative of $f(x) = \sqrt{x}$. The first derivative (found in the previous section) is $f'(x) = \frac{1}{2\sqrt{x}}$. Then

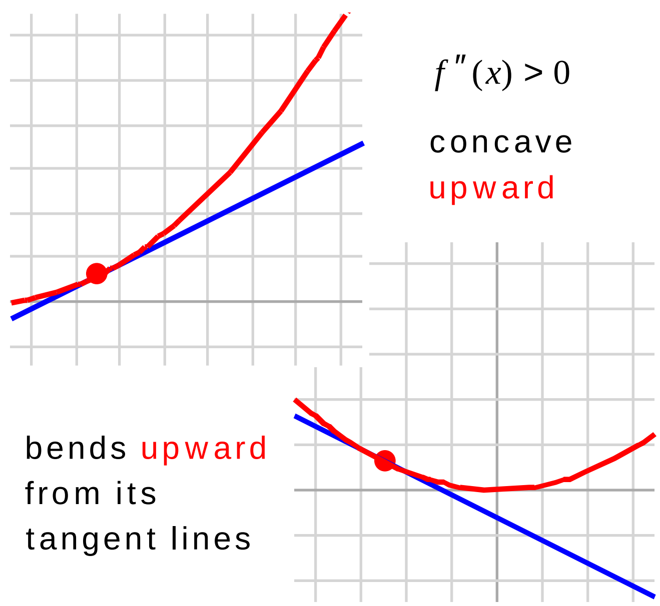
$$
\begin{aligned}
f''(x) &= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \left\{ \frac{1}{2\sqrt{x+h}} - \frac{1}{2\sqrt{x}} \right\} \\
&= \lim_{h \to 0} \frac{1}{2h} \left\{ \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h}\sqrt{x}} \times \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} \right\} \\
&= \lim_{h \to 0} \frac{1}{2h} \left\{ \frac{x - (x+h)}{\sqrt{x+h}\sqrt{x}\left(\sqrt{x} + \sqrt{x+h}\right)} \right\} \\
&= \lim_{h \to 0} \frac{-1}{2\sqrt{x+h}\sqrt{x}\left(\sqrt{x} + \sqrt{x+h}\right)} \\
&= \frac{-1}{2\sqrt{x}\sqrt{x}\left(2\sqrt{x}\right)} \\
&= \frac{-1}{4x\sqrt{x}} \qquad \blacksquare
\end{aligned}
$$

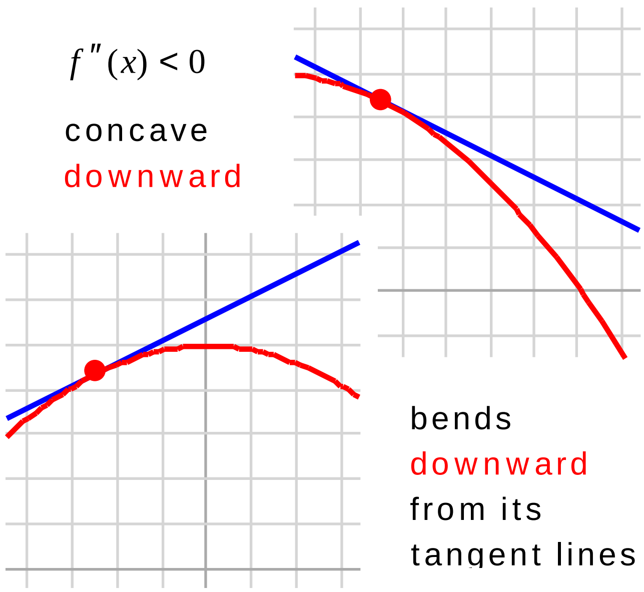
✔ Show that the second derivative of $f(x)=\frac{1}{x}$ is $f''(x)=\frac{2}{x^3}$.

✔ A general quadratic function has the form $q(x)=ax^2+bx+c$ for constants a, b and c with $a\neq 0$. Show that $q'(x)=2ax+b$ and $q''(x)=2a$. What is the third derivative of q? the fourth derivative? the fifth?

Concavity The second derivative of f is the first derivative of f', and so gives the rate of change of the slope of the curve $y=f(x)$. (The function itself may be increasing or decreasing; it's the rate of change of the *slope* that matters here.)

Suppose $f''(x)>0$ on some interval. Then on this interval, the slope of $y=f(x)$ is increasing as x increases, i.e. the curve is getting steeper - if negative, the slope is getting less negative, or if positive, it is getting more positive. The tangent line at any point of the curve has constant slope, so to get steeper, the curve itself must bend upward from its tangent line at any point of the interval- it is ***concave upward***.





Conversely, if $f''(x) < 0$ on an interval, the slope of $y = f(x)$ is decreasing as *x* increases, i.e. the curve is getting less steep as *x* increases- if positive, the slope is getting less positive, or if negative, it is getting more negative. To get less steep, the curve itself must bend downward from its tangent line at any point - it is ***concave downward***.

Example Let's find where the function $f(x) = \frac{1}{6}x^3 - x^2 + 4$ is concave upward or concave downward. We need the second derivtive of *f*. The first derivative is

$$f'(x) = \lim_{h \to 0} \frac{\left\{\frac{1}{6}(x+h)^3 - (x+h)^2 + 4\right\} - \left\{\frac{1}{6}x^3 - x^2 + 4\right\}}{h}$$

$$= \lim_{h \to 0} \frac{\left\{\frac{1}{6}\left(x^3 + 3x^2h + 3xh^2 + h^3\right) - \left(x^2 + 2xh + h^2\right) + 4\right\} - \left\{\frac{1}{6}x^3 - x^2 + 4\right\}}{h}$$

$$= \lim_{h \to 0}\left(\frac{1}{2}x^2 + \frac{1}{2}xh + \frac{1}{6}h^2 - 2x - h\right)$$

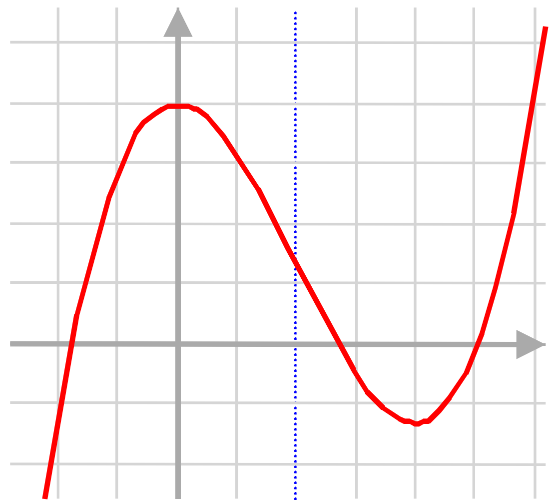
$$= \frac{1}{2}x^2 - 2x$$

expand the cube and the square

and the second derivative is then

$$f''(x) = \lim_{h\to 0} \frac{\left\{\tfrac{1}{2}(x+h)^2 - 2(x+h)\right\} - \left\{\tfrac{1}{2}x^2 - 2x\right\}}{h}$$
$$= \lim_{h\to 0} \frac{\tfrac{1}{2}x^2 + xh + \tfrac{1}{2}h^2 - 2x - 2h - \tfrac{1}{2}x^2 + 2x}{h}$$
$$= \lim_{h\to 0}\left(x + \tfrac{1}{2}h - 2\right)$$
$$= x - 2.$$

Then $f''(x) < 0$ for $x < 2$ and $f''(x) > 0$ for $x > 2$ so f is concave downward for $x < 2$ and concave upward for $x > 2$. The graph of f looks like this:



■

More about notation If $y = f(x)$, then we can write its derivatives as $y' = f'(x)$, $y'' = f''(x)$, . . . , $y^{(n)} = f^{(n)}(x)$. Another equally important notation is the ***Leibniz notation***:

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = y', \quad \frac{d^2y}{dx^2} = \frac{d^2}{dx^2} f(x) = y'', \quad . . . , \quad \frac{d^ny}{dx^n} = \frac{d^n}{dx^n} f(x) = y^{(n)}.$$

The Leibniz notation has the disadvantage that it doesn't directly supply a place to put a value of x where it may be evaluated. This is usually indicated with a vertical line:

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}, \quad f''(a) = \left.\frac{d^2y}{dx^2}\right|_{x=a}, \text{ etc.}$$

Example You showed earlier that for $f(x)=\frac{1}{x}$, $f''(x)=\frac{2}{x^3}$. In other notation, this can be written as

$$y''=\left(\frac{1}{x}\right)'' = \frac{d^2y}{dx^2}=\frac{d^2}{dx^2}\left(\frac{1}{x}\right)=\frac{2}{x^3}.$$

Also $\left.\frac{d^2y}{dx^2}\right|_{x=3}=\frac{2}{27}$. ■

✔ For $y=f(x)=x^2$, explain why $\left.\frac{dy}{dx}\right|_{x=3}\neq\frac{d}{dx}f(3)$.

Acceleration We've seen that the derivative of the position function of an object is the velocity function of that object. The derivative of the velocitiy function gives the rate of change of velocity, i.e. the acceleration of the object.

Defintion. If an object moves along a straight path so that its position relative to some reference point on that path is $p=s(t)$ at time t, then its ***instantaneous acceleration*** at any time t is defined to be $s''(t)$, if this derivative exists.

Example An object moves along the x- axis so that its position relative to the origin at any time t sec. is proportional to t^4+3t. Its initial velocity is 5 units/sec.; we find its acceleration function.

We have that the position function is $x(t)=k\left(t^4+3t\right)$ for some proportionality constant k, so its velocity function is

$$
\begin{aligned}
\frac{dx}{dt} &= \lim_{h\to 0}\frac{x(t+h)-x(t)}{h} \\
&= \lim_{h\to 0}\frac{k\left[(t+h)^4+3(t+h)\right]-k\left[t^4+3t\right]}{h} \\
&= \lim_{h\to 0} k\,\frac{\left[t^4+4t^3h+6t^2h^2+4th^3+h^4+3t+3h\right]-\left[t^4+3t\right]}{h} \\
&= k\lim_{h\to 0}\left(4t^2+6t^2h+4th^2+h^3+3\right) \\
&= k(4t^3+3)
\end{aligned}
$$

expand the fourth power (binomial theorem)

It initial velocity is 5, i.e. $\left.\frac{dx}{dt}\right|_{t=0}=5$, so $k(4\times 0^3+3)=5$, i.e. $k=\frac{5}{3}$. The velocity function is now $\frac{dx}{dt}=\frac{5}{3}\left(4t^3+3\right)$, and its derivative is the acceleration function:

$$
\begin{aligned}
\frac{d^2x}{dt^2} &= \lim_{h\to 0}\frac{\frac{5}{3}\left[4(t+h)^3+3\right]-\frac{5}{3}\left[4t^3+3\right]}{h} \\
&= \lim_{h\to 0}\frac{5}{3}\,\frac{\left[4(t^3+3t^2h+3th^2+h^3+3\right]-\left[4t^3+3\right]}{h} \\
&= \frac{5}{3}\lim_{h\to 0}\left(12t^2+12th+4h^2\right) \\
&= \frac{5}{3}(12t^2) \\
&= 20t^2 \quad \blacksquare
\end{aligned}
$$

3.6 Antiderivatives and Differential Equations

Sometimes we need to go backward from a derivative to the original function.

> **Definition** A function F is an ***antiderivative*** of another function f if $F' = f$.

Example. You showed earlier on that the derivative of $x^2 + x$ is $2x+1$, so one antiderivative of $2x+1$ is $x^2 + x$. Now let's show that $(x+\frac{1}{2})^2$ is also an antiderivative of $2x+1$.

$$\begin{aligned}
\frac{d}{dx}(x+\tfrac{1}{2})^2 &= \lim_{h\to 0}\frac{(x+h+\frac{1}{2})^2-(x+\frac{1}{2})^2}{h} \\
&= \lim_{h\to 0}\frac{\left\{x^2+h^2+\frac{1}{4}+2xh+2(\frac{1}{2})x+2(\frac{1}{2})h\right\}-\left\{x^2+2x(\frac{1}{2})+\frac{1}{4}\right\}}{h} \\
&= \lim_{h\to 0}\{h+2x+1\} \\
&= 2x+1
\end{aligned}$$

Since the derivative of $(x+\frac{1}{2})^2$ is $2x+1$, $2x+1$ is another antiderivative of $(x+\frac{1}{2})^2$. ■

✔ Show that $\frac{1}{3}x^3+5$, $\frac{1}{3}x^3+67$ and $\frac{1}{3}x^3-\pi$ are all antiderivatives of x^2, and find 3 others.

In general, a function has many antiderivatives, but they are all related. To see how, suppose we have two antiderivatives F_1 and F_2 of the same function *f*, i.e. $F_1'(x) = F_2'(x) = f(x)$. Then $F_2'(x) - F_1'(x) = 0$, which says that

$$\lim_{h \to 0} \frac{F_2(x+h) - F_2(x)}{h} - \lim_{h \to 0} \frac{F_1(x+h) - F_1(x)}{h} = 0.$$

Combine the limits and rearrange the numerator :

$$\lim_{h \to 0} \frac{\{F_2(x+h) - F_1(x+h)\} - \{F_2(x) - F_1(x)\}}{h} = 0.$$

This limit is itself a derivative: the derivative of the function $F_2 - F_1$. Since the derivative is always 0, the function $F_2 - F_1$ must be constant: $F_2(x) - F_1(x) = C$ for some constant *C*. This says that $F_2(x) = F_1(x) + C$, i.e.once we have *one* antiderivative of a function, any *other* antiderivative of that function may be found by adding a constant to the first one. If F is any antiderivative of *f*, the ***general antiderivative*** of f is written as $F(x) + C$ for an arbitrary and unspecified constant *C*.

✔ In an earlier example, we found that $x^2 + x$ and $(x + ½)^2$ were both antiderivatives of $2x + 1$. Show that the second is the first plus a constant and find the constant. What is the general antiderivative of $2x + 1$?

There is no general method for calculating the antiderivative of a function (other than recognizing that function as a derivative), but there are many techniques for doing so.

Differential equations A ***differential equation*** is an equation involving a function and its derivatives.

Some examples:

$$5y''' + y'' + 3y' - 2y = \sin x \qquad 4x\, f(x) f''(x) = 1$$

$$\frac{ds}{dt} = 5s + e^t \qquad x^2 y'' - xy' + y = 3x^2$$

A ***solution*** of a differential equation is any function which satisfies it.

Example. Let's show that $y = f(x) = 3x^2 + x$ is a solution of the last equation above. You showed earlier that the first two derivatives of a general quadratic function $q(x) = ax^2 + bx + c$ are $q'(x) = 2ax + b$ and $q''(x) = 2a$. For our particular function, $a = 3$, $b = 1$ and $c = 0$, so we have that $f'(x) = 6x + 1$ and $f''(x) = 6$. Then

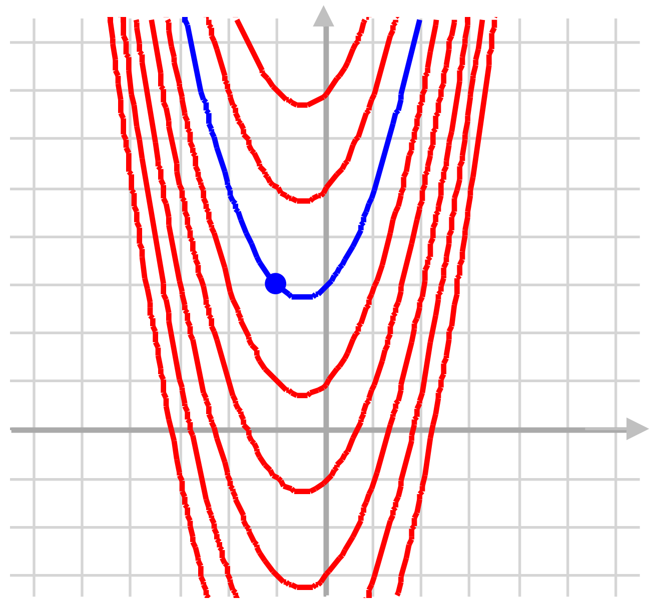
$$x^2 y'' - xy' + y = x^2(6) - x(6x+1) + (3x^2 + x) = 3x^2$$

so $y = 3x^2 + x$ is a solution of $x^2 y'' - xy' + y = 3x^2$. ■

✔ Show that $f(x) = \sqrt{x}$ is a solution of the differential equation

$$4x\, f(x) f''(x) = 1.$$

Differential equations generally have many solutions. The differential equation $y' = 2x + 1$, for example, states that y is an antiderivative of $2x + 1$. We saw earlier that all such y's have the form $y = x^2 + x + C$ for some constant C, i.e. $y = x^2 + x + C$ is the ***general solution*** of $y' = 2x + 1$. In general, the solution of a ***first order differential equation*** (one with only a first order derivative of the function) contains one arbitrary constant, and in most cases, any solution of a first order equation containing an arbitrary constant is the general solution.



The general solution to a differential equations represents a families of curves, with different curves of the family given by different values of the constant(s). The curves in the diagram all have equations of the form $y = x^2 + x + C$ for some constant *C*, and so are all graphs of solutions of $y' = 2x + 1$.

A ***particular solution*** of a differential equation is one with specific values of the constant(s) assigned, so for example, $y = x^2 + x + 3$ is the particular solution of the differential equation $y' = 2x + 1$ which passes through the point $(-1,3)$ - the blue curve in the diagram.

Example. Let's show that $y = 1 + \frac{A}{x}$ is the general solution of the first order differential equation $xy' + y - 1 = 0$, and then find the particular solution that passes through $(1,3)$. We first need y'.

$$
\begin{aligned}
y' &= \lim_{h\to 0}\frac{1}{h}\left\{\left(1+\frac{A}{x+h}\right)-\left(1+\frac{A}{x}\right)\right\} \\
&= \lim_{h\to 0}\frac{1}{h}\left\{\frac{A}{x+h}-\frac{A}{x}\right\} \\
&= \lim_{h\to 0}\frac{A}{h}\left\{\frac{x-(x+h)}{(x+h)x}\right\} \\
&= A\lim_{h\to 0}\frac{-1}{(x+h)x} \\
&= \frac{-A}{x^2}
\end{aligned}
$$

Then

$$
\begin{aligned}
xy'+y-1 &= x\left(\frac{-A}{x^2}\right)+\left(1+\frac{A}{x}\right)-1 \\
&= -\frac{A}{x}+1+\frac{A}{x}-1 \\
&= 0
\end{aligned}
$$

so we have the general solution. For the particular solution, put $(x,y)=(1,3)$ in the general solution: $3=1+\frac{A}{1}$, fron which $A=2$. The particular solution through $(1,3)$ is thus $y=1+\frac{2}{x}$. ■

✔ Show that $y=Bx^3+1$ is the general solution of the first order differential equation $xy'-3y+3=0$. Show that no particular solution passes through the origin, but all pass through $(0,1)$. Find the particular solution which passees through $(-1,0)$, and plot it and several other curves of this family on the same axes. Are there any other points which have no particular solution curve passing thorough them?

Example A basic physical principle states that, on earth, any object falls with a constant acceleration of approximately 9.8 m/sec^2. Let's find a formula for its vertical position.

• We'll need some antiderivatives. Remember that you found the derivatives of a quadratic polynomial earlier:

$$(at^2+bt+c)'=2at+b \quad \text{and} \quad (2at+b)'=2a;$$

the corresponding antiderivatives are all we'll need.

• The derivative of the velocity $v(t)$ is the constant acceleration -9.8 (the sign is negative because we normally measure positive distances upward). Use the second formula with $a=-4.9$ and $b=0$:

$$\{2(-4.9)t\}'=2(-4.9)=-9.8=v'(t).$$

It follows that $v(t)=2(-4.9)t+C$ for some constant *C*. To identify *C*, set $t=0$: $v(0)=C$, i.e., C is the ***initial velocity*** of the falling object (for example, the object could be a projectile fired upward with some positive initial velocity). We have so far

$$v(t)=-9.8t+v(0).$$

• Now let's do the same thing again, since velocity is the derivative of position $s(t)$. Use the first differentation formula with $a=-4.9$, $b=v(0)$ and $c=0$:

$$(-4.9t^2+v(0)t)'=2(-4.9)t+v(0)=s'(t).$$

It follows that $s(t)=-4.9t^2+v(0)t+D$ for some constant *D*. To identify *D*, set $t=0$ again: $s(0)=D$, i.e. D is the ***initial position*** of the object (for example, if the object falls out a window 10 m above ground, *D*=10). Our final formula for the vertical position of a general falling object is

$$s(t) = -4.9t^2 + v(0)t + s(0). \blacksquare$$

Falling body At time t sec., the position of a falling body, measured in meters, is given by

$$s(t) = -4.9t^2 + v(0)t + s(0)$$

where $s(0)$ is its initial position and $v(0)$ is its initial velocity.

3.7 What can go Wrong; Differentiability

Thus far, we have been differentiating functions without worrying about whether the process always works. Since a derivative is a type of limit, in some circumstances, it may not exist.

> **Definition** The function f is ***differentiable*** at $x = a$ whenever $f'(a)$ exists.

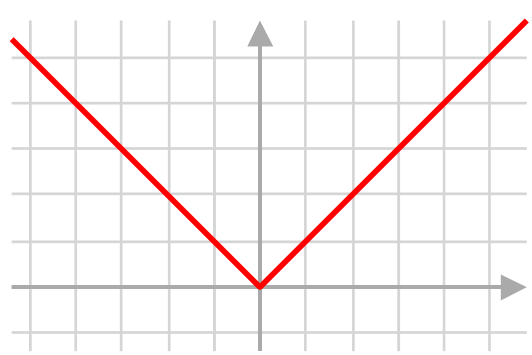
If a function is not differentiable at a point, many different things can go wrong. Let’s look at some of them.

First of all, the curve may simply never straighten out as we zoom in on the point in question.

Example. Let's look at the absolute value function at the point $(0, 0$. Since

$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

the graph of f consists of parts of two straight lines: $y = -x$ for $x < 0$ and $y = x$ for $x \geq 0$. These lines meet at $(0, 0$ and are perpendicular there, and no amount of zooming will ever show otherwise – the graph has a intrinsic “kink” at $(0, 0$.



Let's see how this plays out algebraically. To calculate the derivative of $f(x)=|x|$ at $x=0$, we need to find the limit

$$\lim_{h\to 0}\frac{|0+h|-|0|}{h}=\lim_{h\to 0}\frac{|h|}{h}.$$

We deal with the absolute value by taking separate left- and right- hand limits.

- For $h<0$, $|h|=-h$, so $\lim_{h\to 0}\frac{|h|}{h}=\lim_{h\to 0}(-1)=-1$.
- For $h>0$, $|h|=h$, so $\lim_{h\to 0}\frac{|h|}{h}=\lim_{h\to 0}(+1)=1$.

Since we have different left- and right- hand limits, it follows that the limit we need doesn't exist, so f is not differentiable at $x=0$. ■

✔ Use your grapher to plot the function

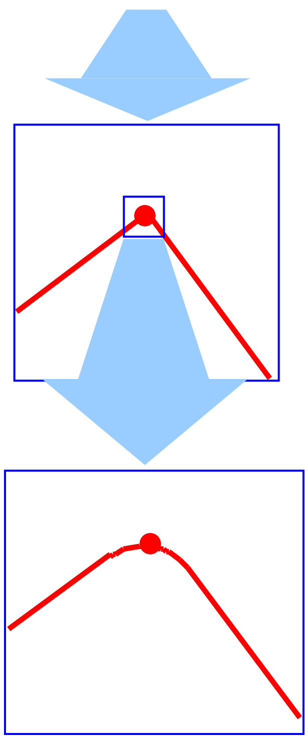
$$f(x)=\left|x^2-x-2\right|.$$

For which x does it appear that f is not differentiable? Verify your guess by finding left- and right- hand limits of the difference quotient at each point.

✔ For most values of the constant m, the function

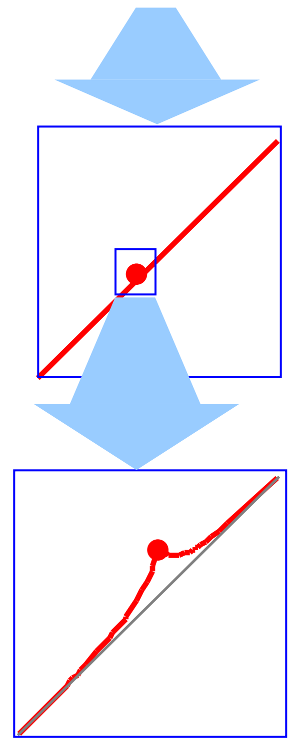
$$f(x)=\begin{cases} 4x-x^2 & \text{for } x\le 0 \\ mx & \text{for } x>0 \end{cases}$$

has a kink at $(0,0$. Use your grapher to plot this function for some value of m, say $m=1$, and then experiment with different values of m to find the one which appears to remove the kink. Once you think you have the correct value for m, verify your guess by showing that f is diffferentiable for this m.



In general, it's not possible to determine for sure if a curve has a kink at some point merely by zooming in on that point with a grapher. No matter how close in to the point you zoom to, it's always possible that, at a higher magnification, what appears to be a kink will "round off",

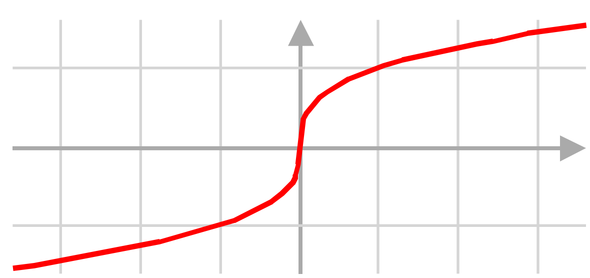
or what appears to be a kink-free curve will develop some minuscule kink hitherto too small to be seen.



Another way a f function can fail to be differentiable at a point a is if its graph "becomes vertical" there:

Example. Consider the function $f(x) = x^{1/3}$. When we try to calculate its derivative at $x = 0$, we get

$$\lim_{h\to 0}\frac{(0+h)^{1/3} - 0^{1/3}}{h} = \lim_{h\to 0}\frac{1}{h^{2/3}} = +\infty$$



so f is not differentiable at $x = 0$. If we plot its graph, we can see what is happening geometrically: the slope of the graph of f becomes infinite at $x = 0$: $\lim_{x\to 0}|f'(x)| = +\infty$. ■

✔ Use your grapher to plot the graph of $f(x) = x^{1/5}\cos x$, and zoom in on the point $(0,0$. What happens to the slope of the curve near this point? Try to calculate the derivative at $x=0$ and see what happens.

In order to be differentiable at a point, a function must first be continuous there.

Theorem If a function is differentiable at a point, then it must be continuous at that point.

Proof. Assume that the function f is differentiable at $x=a$; then we want to show that it is continuous at $x=a$, i.e. that $\lim_{x\to a} f(x) = f(a)$. If we set $h = x - a$, this is the same as showing that

$$\lim_{h\to 0} f(a+h) = f(a).$$

Let's express $f(a+h)$ as a combination of terms:

$$f(a+h) = \left\{\frac{f(a+h)-f(a)}{h}\right\}h + f(a).$$

(Check this by simplifying the right hand side.)

Each of the three parts of the right hand side has a limit:

- ✔ Since f is differentiable at $x=a$, $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(a)$.
- ✔ $\lim_{h\to 0} h = 0$.

✔ Since $f(a)$ is a constant, $\lim_{h\to 0} f(a) = f(a)$.

It follows that

$$\lim_{h\to 0} f(a+h) = f'(a)\times 0 + f(a) = f(a),$$

which is what we wanted to prove. ■

Example We've seen that the function

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is continuous at $x = 0$. Let's see if it is differentiable there. We try to calculate the derivative:

$$\begin{aligned} &\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h\sin\frac{1}{h} - 0}{h} \\ &= \lim_{h\to 0} \sin\frac{1}{h} \end{aligned}$$

Since $\frac{1}{h}$ increases forever as $h \to 0$, sine of it oscillates forever between+1 and -1: the limit doesn't exist. So f is not differentiable at $x = 0$. ■

✔ Use the fact that the function f of this example is continouos at $x = 0$ to show that the functions

$$g_n(x) = \begin{cases} x^n \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

are differentiable at $x = 0$ for any $n > 1$, and find their derivatives there.